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Dressed (renormalized) coordinates in a nonlinear system

G Flores-Hidalgo[1] **and Y W Milla**[2]

[1] Instituto de Física Teorica-IFT/UNESP, Rua Pamplona 145, 01405-900, São Paulo, SP, Brazil
[2] Centro Brasileiro de Pesquisas Fisicas, Rua Dr Xavier Sigaud 150, 22290-180, Rio de Janeiro, RJ, Brazil

E-mail: gflores@ift.unesp.br and yonym@cbpf.br

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Abstract

In previous publications, the concepts of *dressed coordinates* and *dressed states* have been introduced in the context of a harmonic oscillator linearly coupled to an infinity set of other harmonic oscillators. In this paper, we show how to generalize such dressed coordinates and states to a nonlinear version of the mentioned system. Also, we clarify some misunderstandings about the concept of dressed coordinates. Indeed, now we prefer to call them *renormalized coordinates* to emphasize the analogy with the renormalized fields in quantum field theory.

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1. Introduction

In recent works, the concepts of *dressed coordinates* and *dressed states* have been introduced in the context of a harmonic oscillator linearly coupled to an infinite set of other harmonic oscillators [1–5]. As explained in the next section, the introduction of dressed coordinates is necessary in order to give physical consistence to the above system as a model to describe some given physical system. In early works [1–5], the physical system studied through the formalism of dressed coordinates and states was an atom–electromagnetic field system. In this case, the introduction of dressed coordinates and states has shown twofold advantages. From the physical viewpoint, the dressed states behave as expected for the physically measurable states: excited atomic states are unstable whereas the atom in the ground state and no field quanta is stable. On the other hand, it allows exact computations for the probability amplitudes associated with the different radiation processes of the harmonic oscillator [4]. When applied to a confined atom, approximated by the harmonic oscillator, in a spherical cavity of sufficiently small diameter the method accounts for the experimentally observed inhibition of the decaying processes [6, 7].

In [8], an attempt to construct dressed coordinates and dressed states for a nonlinear system has been made. However, the approach used there was more intuitive than formal. The purpose of this paper is to develop a formal method to construct dressed coordinates in nonlinear systems. We will do this by a perturbative expansion in the nonlinear coupling constant. To be specific, we consider the model with Hamiltonian given by

$$H = \frac{1}{2}\left(p_0^2 + \omega_B^2 q_0^2\right) + \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + \omega_k^2 q_k^2 - 2c_k q_k q_0\right) + \sum_{r=0}^{N}\lambda_r \mathcal{T}^{(r)}_{\mu\nu\rho\sigma} q_\mu q_\nu q_\rho q_\sigma + \sum_{r=0}^{N}\alpha_r \mathcal{R}^{(r)}_{\mu\nu\rho\sigma\tau\epsilon} q_\mu q_\nu q_\rho q_\sigma q_\tau q_\epsilon, \tag{1.1}$$

where sums over repeated indices and the limit $N \to \infty$ are understood. In equation (1.1), the bare frequency of the harmonic oscillator, ω_B, is related to the physical frequency, ω_0, by [9, 10]

$$\omega_B^2 = \omega_0^2 + \sum_{k=1}^{N}\frac{c_k^2}{\omega_k^2}. \tag{1.2}$$

The coefficients $\mathcal{T}^{(r)}_{\mu\nu\rho\sigma}$ and $\mathcal{R}^{(r)}_{\mu\nu\rho\sigma\tau\epsilon}$ are chosen in such a way that the Hamiltonian given by equation (1.1) is defined as positive.

In [8], the quartic nonlinear model, $\alpha_r = 0$, has been treated. Here, we will also be mainly interested in the quartic nonlinear model to compare with the early treatment. The sextic nonlinear interaction will be considered only because for some values of λ_r and α_r it is possible to find the exact solution for the ground-state eigenfunction and, as explained at the end of section 3, this fact will permit an exact construction of the dressed coordinates. Then, this sextic model will allow us to test the validity of the strategy developed to obtain the dressed coordinates in a general nonlinear system.

In this paper, we use natural units $c = \hbar = 1$.

2. Defining dressed (renormalized) coordinates and dressed states

The purpose of this section is twofold. First, to make this paper self-contained, we review the concept of dressed coordinates and dressed states as introduced in [1–3]. Second, we clarify some misunderstandings about the concept of dressed coordinates, as we explain below. Indeed, now we prefer to call them *renormalized coordinates* to emphasize that these coordinates are analogous to the renormalized fields in quantum field theory.

In previous works [1–3], dressed coordinates and states have been introduced in the context in which the linear part of Hamiltonian (1.1) is used as a model to describe an atom–electromagnetic field system. Here, we introduce dressed coordinates in the context of a common situation frequently encountered in quantum optics, a field mode inside a cavity interacting with the external modes (to the cavity) of the electromagnetic field [11, 12]. In this case, the oscillator with coordinate q_0 in equation (1.1) represents the field mode inside the cavity and the external field modes are represented by the oscillators with coordinates q_k. If there are no interactions among the cavity field mode and the external ones, the free Hamiltonian obtained from equation (1.1) by setting $c_k = \lambda_r = \alpha_r = 0$ has the following eigenfunctions:

$$\psi_{n_0 n_1 \ldots n_N}(q) \equiv \langle q | n_0, n_1, \ldots, n_N\rangle = \prod_{\mu=0}^{N}\psi_{n_\mu}(q_\mu), \tag{2.1}$$

where $|q\rangle = |q_0, q_1, \ldots, q_N\rangle$ and $\psi_{n_\mu}(q_\mu)$ is the eigenfunction of a harmonic oscillator of frequency ω_μ,

$$\psi_{n_\mu}(q_\mu) = \left(\frac{\omega_\mu}{\pi}\right)^{1/4} \frac{H_{n_\mu}(\sqrt{\omega_\mu} q_\mu)}{\sqrt{2^{n_\mu} n_\mu !}} \, \mathrm{e}^{-\frac{1}{2}\omega_\mu q_\mu^2}. \tag{2.2}$$

The physical meaning of $\psi_{n_0 n_1 \ldots n_N}(q)$ in this case is clear, it represents the state in which there are n_0 photons of frequency ω_0 inside the cavity and n_k photons of frequencies ω_k outside the cavity. Now, let us suppose that at some initial time, that we take as $t = 0$, we prepare the system in the state $\psi_{10\ldots 0}(q)$ in which we have one photon inside the cavity and no photons outside. From experience, it is known that after a time t, the photon inside the cavity decays through the cavity walls, its initial energy inside the cavity being distributed among the external field modes. In other words, the state $\psi_{10\ldots 0}(q)$ is unstable. We can explain this phenomenology by introducing interacting terms between the cavity field mode and the external ones. In our model, described by Hamiltonian (1.1), the interacting terms between the cavity field mode and the external ones are given by the linear and nonlinear couplings of q_0 with q_k, the simplest interacting term being the linear coupling that appears in the second term of the right-hand side of equation (1.1). Obviously, by taking into account these interactions any state of the type $\psi_{n_0 0\ldots 0}(q)$ is rendered unstable, since these states in general are not eigenfunctions of the total interacting Hamiltonian. At this point, we have to mention a problem, the state $\psi_{00\ldots 0}(q)$, that represents the state of no photons inside and outside the cavity is also unstable. This is a serious problem, because it means that photons could be created from nothing, in contradiction with experimental evidence. Obviously, the wrong thing here is the model we are using to describe the physical system. Indeed, in quantum optics the model used to describe the above system, in the case in which only the linear interaction among the inside and outside modes is taken into account, is the rotating wave approximation of Hamiltonian (1.1),

$$H_{\mathrm{rwa}} = \frac{1}{2}\sum_{\mu=0}^{N}\left(p_\mu^2 + \omega_\mu^2 q_\mu^2\right) + \sum_{k=1}^{N}\left(\alpha_k \hat{a}_0^\dagger \hat{a}_k + \alpha_k^* \hat{a}_k^\dagger \hat{a}_0\right), \tag{2.3}$$

where $\hat{a}_\mu$ and $\hat{a}_\mu^\dagger$ are annihilation and creation operators. In this case, the state $\psi_{00\ldots 0}(q)$ is an eigenfunction of H_{rwa} because $\hat{a}_0^\dagger \hat{a}_k \psi_{00\ldots 0}(q) = 0$ and $\hat{a}_k^\dagger \hat{a}_0 \psi_{00\ldots 0}(q) = 0$. Then, if the model with Hamiltonian (2.3) is used there is no problem with the stability of the state $\psi_{00\ldots 0}(q)$. But what if no rotating wave approximation is used? How can we give a physical meaning to the system described by Hamiltonian (1.1) as a model to describe the aforementioned physical situation? The answer lies in the spirit of the renormalization programme in quantum field theory: we maintain the model but redefine what the physical quantities, dynamical or parametrical, must be. In our model, we redefine what the physical coordinates must be. We suppose that the coordinates q_μ that appear in the Hamiltonian given by equation (1.1) are not the physical ones, they are bare coordinates. We introduce renormalized coordinates, q_0' and q_k', respectively for the dressed photons inside and outside the cavity and define them as the physical ones. In previous works, the renormalized coordinates were called dressed coordinates [1–3]; for this reason, from now on we will take these denominations as synonymous. In terms of renormalized coordinates, we introduce dressed states by

$$\begin{aligned}\psi_{n_0 n_1 \ldots n_N}(q') &\equiv \langle q' | n_0, n_1, \ldots, n_N\rangle_d \\ &= \prod_{\mu=0}^{N} \psi_{n_\mu}(q_\mu'),\end{aligned} \tag{2.4}$$

where the subscript d means dressed state, $|q'\rangle = |q'_0, q'_1, \ldots, q'_N\rangle$ and $\psi_{n_\mu}(q'_\mu)$ is given by

$$\psi_{n_\mu}(q'_\mu) = \left(\frac{\omega_\mu}{\pi}\right)^{1/4} \frac{H_{n_\mu}(\sqrt{\omega_\mu} q'_\mu)}{\sqrt{2^{n_\mu} n_\mu!}} \, \mathrm{e}^{-\frac{1}{2}\omega_\mu (q'_\mu)^2}. \tag{2.5}$$

The dressed states given by equation (2.4) are defined as the physically measurable states and describe, in general, n_0 physical photons of frequency ω_0 inside the cavity and n_k physical photons of frequency ω_k outside the cavity. Obviously, in the limit in which the coupling constants c_k, λ_r and α_r vanish the renormalized coordinates q'_μ must approach the bare coordinates q_μ.

As in quantum field theory, the dynamics of the system evolves in Hamiltonian form through the bare coordinates. Then, in order to compute physical quantities, such as decay rates, it will be necessary to obtain the relation between bare and renormalized coordinates. To this end, we use the physical requirement of the stability of the state in which there are no photons inside and outside the cavity. This state is described by the dressed state $\psi_{00\ldots0}(q')$ and this state must be stable if and only if it is defined as one of the eigenfunctions of the interacting Hamiltonian (1.1). Also we require the state $\psi_{00\ldots0}(q')$ to be the one of minimum energy, then we define it as being identical (or proportional) to the ground-state eigenfunction of the Hamiltonian (1.1). Denoting by $\phi_{00\ldots0}(q)$ the ground-state eigenfunction of the Hamiltonian (1.1), then the dressed coordinates must be obtained from

$$\psi_{00\ldots0}(q') \propto \phi_{00\ldots0}(q). \tag{2.6}$$

First, we explicitly construct the dressed coordinates for the linear model obtained from equation (1.1) by setting $\lambda_r = \alpha_r = 0$,

$$H_{\text{linear}} = \frac{1}{2}\left(p_0^2 + \omega_B^2 q_0^2\right) + \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + \omega_k^2 q_k^2 - 2c_k q_k q_0\right). \tag{2.7}$$

Although the task of constructing dressed coordinates in linear systems has been done in preceding works, we repeat here the calculation in order to make this paper self-contained. In the next section, we will consider the nonlinear case. As is well known, the Hamiltonian (2.7) can be diagonalized by means of the introduction of normal coordinates and momenta, Q_r and P_r,

$$q_\mu = \sum_{r=0}^{N} t_\mu^r Q_r, \qquad p_\mu = \sum_{r=0}^{N} t_\mu^r P_r, \qquad \mu = (0, k), \quad k = 1, 2, \ldots, N, \tag{2.8}$$

where $\{t_\mu^r\}$ is an orthogonal matrix whose elements are given by [13, 14]

$$t_0^r = \left[1 + \sum_{k=1}^{N} \frac{c_k^2}{\left(\omega_k^2 - \Omega_r^2\right)^2}\right]^{-\frac{1}{2}}, \qquad t_k^r = \frac{c_k}{\left(\omega_k^2 - \Omega_r^2\right)} t_0^r. \tag{2.9}$$

In normal coordinates, the Hamiltonian (2.7) reads

$$H_{\text{linear}} = \frac{1}{2}\sum_{r=0}^{N}\left(P_r^2 + \Omega_r^2 Q_r^2\right), \tag{2.10}$$

where the Ω_r are the normal frequencies, corresponding to the collective modes and given as solutions of [13, 14]

$$\omega_0^2 - \Omega_r^2 = \sum_{k=1}^{N} \frac{c_k^2 \Omega_r^2}{\omega_k^2\left(\omega_k^2 - \Omega_r^2\right)}. \tag{2.11}$$

The eigenfunctions of the Hamiltonian (2.10) are given by

$$\begin{aligned}\phi_{n_0 n_1 \ldots n_N}(Q) &\equiv \langle Q | n_0, n_1, \ldots, n_N \rangle_c \\ &= \prod_{r=0}^{N} \phi_{n_r}(Q_r),\end{aligned} \tag{2.12}$$

where the subscript c means collective state, $|Q\rangle = |Q_0, Q_1, \ldots, Q_N\rangle$ and $\phi_{n_r}(Q_r)$ are the wavefunctions corresponding to one-dimensional harmonic oscillators of frequencies Ω_r,

$$\phi_{n_r}(Q_r) = \left(\frac{\Omega_r}{\pi}\right)^{1/4} \frac{H_{n_r}(\sqrt{\Omega_r} Q_r)}{\sqrt{2^{n_r} n_r!}} \, \mathrm{e}^{-\frac{1}{2}\Omega_r Q_r^2}. \tag{2.13}$$

Now, the dressed coordinates are obtained from the condition given by equation (2.6) that in terms of normal coordinates can be written as $\psi_{00\ldots0}(q') \propto \phi_{00\ldots0}(Q)$. Then, using equations (2.4) and (2.12) in the above relation, we have

$$\exp\left(-\frac{1}{2}\sum_{\mu=0}^{N} \omega_\mu (q'_\mu)^2\right) \propto \exp\left(-\frac{1}{2}\sum_{r=0}^{N} \Omega_r Q_r^2\right), \tag{2.14}$$

from which we obtain

$$q'_\mu = \sum_{r=0}^{N} \sqrt{\frac{\Omega_r}{\omega_\mu}} t_\mu^r Q_r, \tag{2.15}$$

as can be seen by direct substitution in equation (2.14) and using the orthonormality property of the matrix $\{t_\mu^r\}$. Here, we have to remark that in principle any orthogonal matrix $\{\eta_\mu^r\}$ can be used in equation (2.15) in order to accomplish equation (2.14), but because in the limit $c_k \to 0$ the dressed coordinates must approach the bare ones it is natural to choose the matrix $\{t_\mu^r\}$ in equation (2.15), since in the limit $c_k \to 0$, $t_\mu^r \to \delta_\mu^r$. Also, we would like to stress that our dressed coordinates and states are not the same as the ones called by the same name in other references [15–19] where the authors called dressed coordinates and states, respectively, the collective coordinates and states.

The introduction of the renormalized coordinates through equation (2.15) guarantees the stability of the dressed state $\psi_{00\ldots0}(q')$; however, since the other dressed states are not energy eigenfunctions, they will not remain stable. For example, $\psi_{10\ldots0}(q')$, the state in which there is one photon inside the cavity and no photons outside, will decay to a state in which there is no photon inside the cavity but there is a photon of some given frequency outside the cavity, $\psi_{00\ldots01_k0\ldots0}(q')$. Then we see that our dressed coordinates are useful to describe the decay of an initial photon inside the cavity through the cavity walls.

We have to remark that the dressed coordinates here introduced are not a simple change of variables, they are new coordinates in their own right and are introduced by the physical consistence requirement of the model. Equation (2.15) cannot be seen as a coordinate transformation, the coordinates q'_μ and q_μ do not represent, geometrically, the same point in configuration space. The situation is analogous to what happens in quantum field theory. In this case, not only the dynamical variables, the fields, are redefined but also the parametrical ones, such as the masses and coupling constants. At this point, we have to call attention to the fact that in expression (1.1) a frequency renormalization has already been performed. As can be seen in equation (1.2), the frequency of the photon inside the cavity has already been renormalized. Then, in the linear model given by equation (2.7) we have renormalized the frequency of the photon inside the cavity and the coordinates of the field modes inside and outside the cavity. When considering the nonlinear terms nothing guarantees that only these renormalizations will be sufficient. Perhaps it will be necessary to renormalize the frequencies of the external modes and the coupling constants. Also the cavity mode frequency renormalization, equation (1.2) and the renormalized coordinates could be modified. In particular, in the next section we address the question of how the relation given by equation (2.15) is changed when nonlinear terms are taken into account.

As already mentioned, in previous works [1–3], dressed coordinates and states have been introduced in the context in which the system with Hamiltonian (2.7) is used as an oversimplified model to describe an atom–electromagnetic field system. In this case, the harmonic oscillator with index zero represents the bare atom and the other oscillators represent the bare electromagnetic field modes. On the other hand, the renormalized coordinates q_0' and q_k' are defined, respectively, as the physical coordinates of the atom and electromagnetic field modes. The dressed state $\psi_{n_0,n_1,\ldots,n_N}(q')$ represents the state in which the atom is in the n_0th excited level and there are n_k photons of frequencies ω_k. The state $\psi_{00\ldots0}(q')$, in which the atom is in its ground state and there are no photons, according to experience, must be stable. The stability of $\psi_{00\ldots0}(q')$ is guaranteed if one introduces the renormalized coordinates in the way we have described above. Then, we have a unified way to study quite different physical systems and because q_0' could be the coordinate of an (oversimplified) atom or the coordinate of the electromagnetic field mode inside a cavity, from now on we will refer to the harmonic oscillator with index zero simply as the particle oscillator.

Before leaving this section, it will be useful to establish the relation between $\psi_{n_0n_1\ldots n_N}(q') = \langle q'|n_0, n_1, \ldots, n_N\rangle_d$ and $\langle Q|n_0, n_1, \ldots, n_N\rangle_d$. To this end, we write

$$
\begin{aligned}
{}_d\langle n_0, n_1, \ldots, n_N|m_0, m_1, \ldots, m_N\rangle_d &= \int \mathrm{d}q'\ {}_d\langle n_0, n_1, \ldots, n_N|q'\rangle\langle q'|m_0, m_1, \ldots, m_N\rangle_d \\
&= \int \mathrm{d}Q \left|\frac{\partial q'}{\partial Q}\right|_d \langle n_0, n_1, \ldots, n_N|q'\rangle\langle q'|m_0, m_1, \ldots, m_N\rangle_d \\
&= \int \mathrm{d}Q\ {}_d\langle n_0, n_1, \ldots, n_N|Q\rangle\langle Q|m_0, m_1, \ldots, m_N\rangle_d, \qquad (2.16)
\end{aligned}
$$

where $\mathrm{d}q' = \prod_{\mu=0}^{N} \mathrm{d}q'_\mu$, $\mathrm{d}Q = \prod_{r=0}^{N} \mathrm{d}Q_r$ and $|\partial q'/\partial Q|$ is the Jacobian associated with the transformation $q'_\mu \to Q_r$. From equation (2.16), we get

$$
\langle Q|n_0, n_1, \ldots, n_N\rangle_d = \left|\frac{\partial q'}{\partial Q}\right|^{1/2} \langle q'|n_0, n_1, \ldots, n_N\rangle_d. \qquad (2.17)
$$

Taking $n_0 = n_1 = \cdots = n_N = 0$ in equation (2.17) and using $\psi_{00\ldots0}(q') \propto \phi_{00\ldots0}(Q)$, we get

$$
|0, 0, \ldots, 0\rangle_d \propto \int \mathrm{d}Q \left|\frac{\partial q'}{\partial Q}\right|^{1/2} |Q\rangle\langle Q|0, 0, \ldots, 0\rangle_c. \qquad (2.18)
$$

In the linear case, we easily get, from equation (2.15), $|\partial q'/\partial Q| = \prod_{r,\mu=0}^{N} \sqrt{\Omega_r/\omega_\mu}$ and using this result in equation (2.18), we obtain

$$
|0, 0, \ldots, 0\rangle_d \propto |0, 0, \ldots, 0\rangle_c. \qquad (2.19)
$$

For a nonlinear system, certainly a relation of the type given by equation (2.19) will not hold.

In the next section, we construct dressed coordinates in the nonlinear model described by the Hamiltonian given in equation (1.1).

3. Constructing renormalized coordinates in a nonlinear model

Now we are ready to construct dressed coordinates and dressed states in the nonlinear model with Hamiltonian given by equation (1.1). For this purpose, we have to find first, the eigenfunctions of this Hamiltonian, in particular its ground-state eigenfunction. In order to maintain things as simple as possible and to compare with the early treatment given to the problem in [8], we consider the nonlinear quartic interaction obtained from the model with Hamiltonian (1.1) by setting $\alpha_r = 0$. Following [8], we make the simplest choice for the coefficients $\mathcal{T}^{(r)}_{\mu\nu\rho\sigma}$,

$$\mathcal{T}^{(r)}_{\mu\nu\rho\sigma} = t^r_\mu t^r_\nu t^r_\rho t^r_\sigma. \tag{3.1}$$

Substituting equations (2.8) and (3.1) into equation (1.1), we get

$$H = \frac{1}{2}\sum_{r=0}^{N}\left(P_r^2 + \Omega_r^2 Q_r^2 + 2\lambda_r Q_r^4\right), \tag{3.2}$$

that is, we obtain a system of uncoupled quartic anharmonic oscillators. In equation (3.2), it can be seen that λ_r has dimension [*frequency*]^3. Then we can write $\lambda_r = \lambda\Omega_r^3$, where λ is a dimensionless coupling constant. The eigenfunctions of the Hamiltonian (3.2) can be written as

$$\begin{aligned}\phi_{n_0 n_1\ldots n_N}(Q;\lambda) &\equiv \langle Q|n_0, n_1,\ldots,n_N;\lambda\rangle_c \\ &= \prod_{r=0}^{N}\phi_{n_r}(Q_r;\lambda),\end{aligned} \tag{3.3}$$

where $\phi_{n_r}(Q_r;\lambda)$ are eigenfunctions of $\left(P_r^2 + \Omega_r^2 Q_r^2 + 2\lambda\Omega_r^3 Q_r^4\right)/2$ and can be written formally as (see the appendix)

$$\phi_{n_r}(Q_r;\lambda) = \left(\frac{\Omega_r}{\pi}\right)^{1/4}\left[\frac{H_{n_r}(\sqrt{\Omega_r}Q_r)}{\sqrt{2^{n_r}n_r!}} + \sum_{l=1}^{\infty}\lambda^l G^{(l)}_{n_r}(\sqrt{\Omega_r}Q_r)\right]\mathrm{e}^{-\frac{\Omega_r}{2}Q_r^2}, \tag{3.4}$$

where $G^{(l)}_{n_r}(\sqrt{\Omega_r}Q_r)$ are linear combinations of Hermite polynomials. The corresponding eigenvalues of the Hamiltonian (3.2) are given by

$$E_{n_0 n_1\ldots n_N}(\lambda) = \sum_{r=0}^{N} E_{n_r}(\lambda), \tag{3.5}$$

where $E_{n_r}(\lambda)$ are the eigenvalues corresponding to the eigenstates given in equation (3.4),

$$E_{n_r}(\lambda) = \left(n_r + \frac{1}{2}\right)\Omega_r + \sum_{l=1}^{\infty}\lambda^l E^{(l)}_{n_r}, \tag{3.6}$$

with the $E^{(l)}_{n_r}$ obtained by using standard perturbation theory (see the appendix).

Taking $n_0 = n_1 = \cdots = n_N = 0$ in equation (3.3), we get for the ground-state eigenfunction of the total system,

$$\phi_{00\ldots0}(Q;\lambda) = \prod_{r=0}^{N}\left(\frac{\Omega_r}{\pi}\right)^{1/4}\left[1 + \sum_{l=1}^{\infty}\lambda^l G^{(l)}_0(\sqrt{\Omega_r}Q_r)\right]\mathrm{e}^{-\frac{\Omega_r}{2}Q_r^2}. \tag{3.7}$$

To properly define the dressed coordinates it is convenient to write the above equation as

$$\begin{aligned}\phi_{00\ldots0}(Q;\lambda) &= \prod_{r=0}^{N}\left(\frac{\Omega_r}{\pi}\right)^{1/4}\left[1 + \sum_{l=1}^{\infty}\lambda^l G^{(l)}_0(0) + \sum_{l=1}^{\infty}\lambda^l\left(G^{(l)}_0(\sqrt{\Omega_r}Q_r) - G^{(l)}_0(0)\right)\right]\mathrm{e}^{-\frac{\Omega_r}{2}Q_r^2} \\ &\propto \prod_{r=0}^{N}\left[1 + \sum_{n=0}^{\infty}(-1)^n\sum_{l_0 l_1\ldots l_n=1}^{\infty}\lambda^{l_0+l_1+\cdots+l_n}\left(G^{(l_0)}_0(\sqrt{\Omega_r}Q_r) - G^{(l_0)}_0(0)\right)G^{(l_1)}_0(0)\ldots G^{(l_n)}_0(0)\right]\mathrm{e}^{-\frac{\Omega_r}{2}Q_r^2},\end{aligned} \tag{3.8}$$

where in the second line we factored the term $1 + \sum_{l=1}^{\infty} \lambda^l G_0^{(l)}(0)$ and used $(1 + x)^{-1} = \sum_{n=0}^{\infty}(-1)^n x^n$.

The physically measurable states, the dressed states, are defined by equations (2.4)–(2.5). Hence, the dressed coordinates q'_μ must be obtained from equation (2.6) that can be written as $\psi_{00\ldots0}(q') \propto \phi_{00\ldots0}(Q;\lambda)$, which by using equations (2.4), (2.5) and (3.8) can be written as

$$\exp\left(-\frac{1}{2}\sum_{\mu=0}^{N}\omega_\mu (q'_\mu)^2\right) = \prod_{r=0}^{N}\left[1 + \sum_{n=0}^{\infty}(-1)^n \sum_{l_0 l_1\ldots l_n=1}^{\infty} \lambda^{l_0+l_1+\cdots+l_n}\left(G_0^{(l_0)}(\sqrt{\Omega_r}Q_r) - G_0^{(l_0)}(0)\right)G_0^{(l_1)}(0)\ldots G_0^{(l_n)}(0)\right]\mathrm{e}^{-\frac{\Omega_r}{2}Q_r^2}. \tag{3.9}$$

Now, we write a perturbative expansion in λ for q'_μ,

$$q'_\mu = \sum_{r=0}^{N}\sqrt{\frac{\Omega_r}{\omega_\mu}}t_\mu^r\left[Q_r + \frac{1}{\sqrt{\Omega_r}}\sum_{l=1}^{\infty}\lambda^l F_r^{(l)}(\sqrt{\Omega_r}Q_r)\right]. \tag{3.10}$$

Replacing equation (3.10) in equation (3.9) and using the orthonormality property of the matrix $\{t_\mu^r\}$, we get

$$\exp\left[-\sum_{l=1}^{\infty}\lambda^l\sqrt{\Omega_r}Q_r F_r^{(l)}(\sqrt{\Omega_r}Q_r) - \frac{1}{2}\sum_{l,m=1}^{\infty}\lambda^{l+m}F_r^{(l)}(\sqrt{\Omega_r}Q_r)F_r^{(m)}(\sqrt{\Omega_r}Q_r)\right] = 1 + \sum_{n=0}^{\infty}(-1)^n\sum_{l_0 l_1\ldots l_n=1}^{\infty}\lambda^{l_0+l_1+\cdots+l_n}\left(G_0^{(l_0)}(\sqrt{\Omega_r}Q_r) - G_0^{(l_0)}(0)\right)G_0^{(l_1)}(0)\ldots G_0^{(l_n)}(0). \tag{3.11}$$

Expanding the exponential in the left-hand side of equation (3.11) and identifying equal powers of λ, we can obtain all the $F_r^{(l)}(\sqrt{\Omega_r}Q)$. The general expression is very complicated, here we only write the first three terms:

$$F_r^{(1)}(\xi_r) = -\frac{1}{\xi_r}\left(G_0^{(1)}(\xi_r) - G_0^{(1)}(0)\right), \tag{3.12}$$

$$F_r^{(2)}(\xi_r) = -\frac{1}{\xi_r}\left[\frac{1}{2}(1-\xi_r^2)\left(F_r^{(1)}(\xi_r)\right)^2 + G_0^{(1)}(0)\xi_r F_r^{(1)}(\xi_r) + G_0^{(2)}(\xi_r) - G_0^{(2)}(0)\right] \tag{3.13}$$

and

$$F_r^{(3)}(\xi_r) = -\frac{1}{\xi_r}\left[(1-\xi_r^2)F_r^{(1)}(\xi_r)F_r^{(2)}(\xi_r) + \xi_r\left(\frac{\xi_r^2}{3!} - \frac{1}{2}\right)\left(F_r^{(1)}(\xi_r)\right)^3 + \xi_r F_r^{(1)}(\xi_r)\left(G_0^{(2)}(0) - \left(G_0^{(1)}(0)\right)^2\right) - \left(G_0^{(2)}(\xi_r) - G_0^{(2)}(0)\right)G_0^{(1)}(0) + G_0^{(3)}(\xi_r) - G_0^{(3)}(0)\right], \tag{3.14}$$

where $\xi_r = \sqrt{\Omega_r}Q_r$. From the appendix, using equations (A.11)–(A.14) in equations (3.12) and (3.13) we get, respectively,

$$F_r^{(1)}(\xi_r) = \tfrac{1}{4}\left(3\xi_r + \xi_r^3\right) \tag{3.15}$$

and

$$F_r^{(2)}(\xi_r) = -\tfrac{1}{16}\left(\tfrac{93}{2}\xi_r + 14\xi_r^3 + \tfrac{11}{6}\xi_r^5\right). \tag{3.16}$$

Replacing the above equations in equation (3.10), we obtain, at order λ^2,

$$\xi'_\mu = \sum_{r=0}^{N} t_\mu^r \left[\xi_r + \frac{\lambda}{4}(3\xi_r + \xi_r^3) - \frac{\lambda^2}{16}\left(\frac{93}{2}\xi_r + 14\xi_r^3 + \frac{11}{6}\xi_r^5\right)\right] + \mathcal{O}(\lambda^3), \tag{3.17}$$

where we have introduced the dimensionless dressed coordinate $\xi'_\mu = \sqrt{\omega_\mu} q'_\mu$.

Before concluding this section we would like to comment about why we factored the term $1 + \sum_{l=1}^{\infty} \lambda^l G_0^{(l)}(0)$ in equation (3.8). Note that we define the dressed coordinates q'_μ by means of the proportionality $\psi_{00\ldots0}(q') \propto \phi_{00\ldots0}(Q;\lambda)$. To promote this proportionality into an equality we have to take care in order to obtain a well-behaved relation between dressed and collective coordinates, for example, it would be undesirable for any singular relation. To see how the above undesirable situation could happen, define the dressed coordinates through equation (3.7) without the factorization of the term $1 + \sum_{l=1}^{\infty} \lambda^l G_0^{(l)}(0)$. It is easy to show that in this case the relation between q'_μ and Q_r is singular. For example, computing $F_r^1(\xi_r)$ and $F_r^2(\xi_r)$ it is obtained that

$$F_r^{(1)}(\xi_r) = -\frac{1}{\xi_r} G_0^{(1)}(\xi_r), \tag{3.18}$$

$$F_r^{(2)}(\xi_r) = -\frac{1}{\xi_r}\left[\frac{1}{2}\left(1 - \xi_r^2\right)\left(F_r^{(1)}(\xi_r)\right)^2 + G_0^{(2)}(\xi_r)\right]. \tag{3.19}$$

Since $G_0^{(1)}(\xi_r)$ and $G_0^{(2)}(\xi_r)$ are nonhomogeneous functions of ξ_r (see the appendix, equations (A.11) and (A.12)) then equations (3.18) and (3.19) are singular in $\xi_r = 0$. Consequently, the dressed coordinates defined through this prescription are not well defined. To understand what is happening and how to remedy the problem, note that such singularity means that $\xi_r F_r^{(l)}(\xi_r)$ is nonhomogeneous in ξ_r. But the effect of this nonhomogeneous term on the wavefunction (that contains terms of the type $\mathrm{e}^{-\lambda^l \xi_r F_r^{(l)}(\xi_r)}$, see equation (3.11)) is just equal to a constant factorization term. Then, to remedy the problem we have to make a convenient factorization in $\psi_{00..0}(q')$ or, equivalently, in $\phi_{00..0}(Q)$ before promoting the proportionality into an equality. That our choice, the factorization of $1 + \sum_{l=1}^{\infty} \lambda^l G_0^{(l)}(0)$ in equation (3.8), is the correct one is supported by the fact that we have obtained a well-behaved relation between dressed and normal coordinates. To further support our choice, we consider a system in which we can solve exactly for the ground-state eigenfunction of the system, allowing us to obtain exact dressed coordinates. Comparing these exact dressed coordinates and the perturbative ones, we have to get the same answer. The model is the one whose Hamiltonian is given by equation (1.1) with coupling constants defined by

$$\lambda_r \mathcal{T}^{(r)}_{\mu\nu\rho\sigma} = \frac{\lambda \Omega_r^3}{(1 - 3\lambda)^{3/2}} t_\mu^r t_\nu^r t_\rho^r t_\sigma^r, \qquad \alpha_r \mathcal{R}^{(r)}_{\mu\nu\rho\sigma\tau\epsilon} = \frac{\lambda^2 \Omega_r^4}{2(1 - 3\lambda)^2} t_\mu^r t_\nu^r t_\rho^r t_\sigma^r t_\tau^r t_\epsilon^r. \tag{3.20}$$

Using the above expression in equation (1.1), we get a system of uncoupled sextic anharmonic oscillators,

$$H = \frac{1}{2}\sum_{r=0}^{N}\left(P_r^2 + \Omega_r^2 Q_r^2 + \frac{2\lambda\Omega_r^3}{(1 - 3\lambda)^{3/2}} Q_r^4 + \frac{\lambda^2 \Omega_r^4}{(1 - 3\lambda)^2} Q_r^6\right). \tag{3.21}$$

By direct substitution it is easy to show that the above Hamiltonian as ground-state eigenfunction has the following solution [20]:

$$\phi(Q;\lambda) = \mathcal{N} \exp\left(-\sum_{r=0}^{N}\left(\beta_r Q_r^2 + \lambda \beta_r^2 Q_r^4\right)\right), \tag{3.22}$$

where $\mathcal{N}$ is a normalization constant,

$$\beta_r = \frac{\Omega_r}{2\sqrt{1-3\lambda}} \tag{3.23}$$

and the corresponding ground-state energy is given by

$$E(\lambda) = \sum_{r=0}^{N} \beta_r. \tag{3.24}$$

Now, the dressed coordinates can be defined by

$$\exp\left(-\sum_{\mu=0}^{N} \omega_\mu (q'_\mu)^2\right) = \exp\left(-\sum_{r=0}^{N}\left(\beta_r Q_r^2 + \lambda\beta_r^2 Q_r^4\right)\right), \tag{3.25}$$

from which we obtain

$$\xi'_\mu = \sum_{r=0}^{N} t_\mu^r \xi_r \left(\frac{1}{\sqrt{1-3\lambda}} + \frac{\lambda\xi_r^2}{2(1-3\lambda)}\right)^{1/2}. \tag{3.26}$$

Note that at order λ both the quartic and sextic anharmonic Hamiltonians, given respectively by equations (3.2) and (3.21), are equivalent. Then, if our strategy to define the dressed coordinates perturbatively is the correct one, at order λ equation (3.17) must be identical to equation (3.26). Expanding equation (3.26) at order λ, we can see that it is indeed the case. Then we conclude that our strategy to construct the dressed coordinates perturbatively is the correct one.

4. The decay process of the first excited state

In [8], the probability of the particle oscillator remaining in the first excited state has been computed at first order for the nonlinear quartic interaction. However, as we have already mentioned, the approach used there was more intuitive than formal. In order to see to what extent such a calculation is correct, in this section we compute the same quantity by using the formalism presented in the last section. To maintain reasoning as general as possible, we present the steps necessary to compute the probability amplitude associated with the most general transition,

$$\mathcal{A}_{n_0 n_1 \ldots n_N}^{m_0 m_1 \ldots m_N}(t) = {}_d\langle m_0, m_1, \ldots, m_N|\,\mathrm{e}^{-\mathrm{i}Ht}|n_0, n_1, \ldots, n_N\rangle_d, \tag{4.1}$$

that is, if we prepare the system initially at time $t = 0$ in the dressed state $|n_0, n_1, \ldots, n_N\rangle_d$, then equation (4.1) gives the probability amplitude of finding, in a measurement performed at time t, the dressed state $|m_0, m_1, \ldots, m_N\rangle_d$. Introducing a complete set of eigenstates of the total Hamiltonian H, given by equation (3.3), in equation (4.1) we find

$$\begin{aligned}
&\mathcal{A}_{n_0 n_1 \ldots n_N}^{m_0 m_1 \ldots m_N}(t) \\
&= \sum_{l_0 l_1 \ldots l_N = 0}^{\infty} {}_d\langle m_0, m_1, \ldots, m_N|\,\mathrm{e}^{-\mathrm{i}Ht}|l_0, l_1, \ldots, l_N; \lambda\rangle_c\, {}_c\langle l_0, l_1, \ldots, l_N; \lambda|n_0, n_1, \ldots, n_N\rangle_d \\
&= \sum_{l_0 l_1 \ldots l_N = 0}^{\infty} T_{n_0 n_1 \ldots n_N}^{l_0 l_1 \ldots l_N}(\lambda)\, T_{m_0 m_1 \ldots m_N}^{l_0 l_1 \ldots l_N}(\lambda)\, \mathrm{e}^{-\mathrm{i}t E_{l_0 l_1 \ldots l_N}(\lambda)},
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
T_{n_0 n_1 \ldots n_N}^{l_0 l_1 \ldots l_N}(\lambda) &= \int \mathrm{d}Q\, {}_c\langle l_0, l_1, \ldots, l_N; \lambda|Q\rangle\langle Q|n_0, n_1, \ldots, n_N\rangle_d \\
&= \int \mathrm{d}Q \left|\frac{\partial q'}{\partial Q}\right|^{1/2} \phi_{l_0 l_1 \ldots l_N}(Q;\lambda)\,\psi_{n_0 n_1 \ldots n_N}(q').
\end{aligned} \tag{4.3}$$

In the second line of the above expression, we have used equation (2.17). From equation (3.10) we easily get the Jacobian $|\partial q'/\partial Q|$,

$$\left|\frac{\partial q'}{\partial Q}\right| = \prod_{r,\mu=0}^{N}\left|\sqrt{\frac{\Omega_r}{\omega_\mu}}\left(1+\frac{1}{\sqrt{\Omega_r}}\sum_{l=1}^{\infty}\lambda^l\frac{\partial}{\partial Q_r}F_r^{(l)}(\sqrt{\Omega_r}Q_r)\right)\right|. \tag{4.4}$$

Now we evaluate, at first order in λ, the probability amplitude for the particle oscillator to remain at time t in the first excited state if it has been prepared in that state at time $t=0$. This quantity is obtained by taking $n_0=m_0=1$ and $n_k=m_k=0$ in equation (4.2),

$$\mathcal{A}_{10\ldots0}^{10\ldots0}(t) = \sum_{l_0l_1\ldots l_N=0}^{\infty}\left[T_{10\ldots0}^{l_0l_1\ldots l_N}(\lambda)\right]^2 \mathrm{e}^{-\mathrm{i}tE_{l_0l_1\ldots l_N}(\lambda)}. \tag{4.5}$$

Note that to compute $\mathcal{A}_{10\ldots0}^{10\ldots0}(t)$ to first order in λ we have to find $T_{10\ldots0}^{l_0l_1\ldots l_N}(\lambda)$, defined in equation (4.3), at order λ. Replacing equation (3.15) in equation (4.4), we get

$$\left|\frac{\partial q'}{\partial Q}\right| = \left(\prod_{\mu,r=0}^{N}\frac{\Omega_r}{\omega_\mu}\right)^{1/4}\left[1+\frac{3\lambda}{32}\sum_{s=0}^{N}(6H_0(\sqrt{\Omega_s}Q_s)+H_2(\sqrt{\Omega_s}Q_s)\right]+\mathcal{O}(\lambda^2). \tag{4.6}$$

At order λ, from equation (3.3), we have, for $\phi_{l_0l_1\ldots l_N}(Q,\lambda)$,

$$\phi_{l_0l_1\ldots l_N}(Q,\lambda) = \prod_{r=0}^{N}\phi_{l_r}(Q_r)+\lambda\sum_{r=0}^{N}\left[\left(\frac{\Omega_r}{\pi}\right)^{1/4}G_{l_r}^{(1)}(\sqrt{\Omega_r}Q_r)\,\mathrm{e}^{-\frac{\Omega_r}{2}Q_r^2}\prod_{s\neq r}\phi_{l_s}(Q_s)\right]+\mathcal{O}(\lambda^2), \tag{4.7}$$

where $\phi_{l_r}(Q_r)$ is given by equation (2.13),

$$\begin{aligned}G_{l_r}^{(1)}(\sqrt{\Omega_r}Q_r) &= a_{l_r}H_{l_r-4}(\sqrt{\Omega_r}Q_r)+b_{l_r}H_{l_r-2}(\sqrt{\Omega_r}Q_r)\\ &\quad+c_{l_r}H_{l_r+2}(\sqrt{\Omega_r}Q_r)+d_{l_r}H_{l_r+4}(\sqrt{\Omega_r}Q_r),\end{aligned} \tag{4.8}$$

and a_{l_r}, b_{l_r}, c_{l_r} and d_{l_r} are given in the appendix, equation (A.13). Using equation (3.9) and equation (3.10), we have for $\psi_{10\ldots0}(q')$,

$$\begin{aligned}\psi_{10\ldots0}(q') &= \left(\prod_{\mu=0}^{N}\frac{\omega_\mu}{\pi}\right)^{1/4}\frac{H_1(\sqrt{\omega_0}q_0')}{\sqrt{2}}\exp\left(-\frac{1}{2}\sum_{\mu=0}^{N}\omega_\mu(q_\mu')^2\right)\\ &= \left(\prod_{\mu=0}^{N}\frac{\omega_\mu}{\pi}\right)^{1/4}\sum_{r=0}^{N}\frac{t_0^r}{\sqrt{2}}\Bigg[H_1(\sqrt{\Omega_r}Q_r)+2\lambda F_r^{(1)}(\sqrt{\Omega_r}Q_r)-\lambda H_1(\sqrt{\Omega_r}Q_r)\\ &\quad\times\sum_{s=0}^{N}\sqrt{\Omega_s}Q_sF_s^{(1)}(\sqrt{\Omega_s}Q_s)\Bigg]\exp\left(-\frac{1}{2}\sum_{u=0}^{N}\Omega_uQ_u^2\right)+\mathcal{O}(\lambda^2).\end{aligned} \tag{4.9}$$

Replacing equations (4.6), (4.7) and (4.9) in equation (4.3), we obtain after a long, but straightforward, calculation

$$\begin{aligned}T_{10\ldots0}^{l_0l_1\ldots l_N}(\lambda) &= \sum_{r=0}^{N}t_0^r\delta_{l_r1}\prod_{s\neq r}\delta_{l_s0}+9\frac{\sqrt{6}}{16}\lambda\sum_{r=0}^{N}t_0^r\delta_{l_r3}\prod_{s\neq r}\delta_{l_s0}\\ &\quad+3\frac{\sqrt{2}}{16}\lambda\sum_{r\neq s}t_0^r\delta_{l_r1}\delta_{l_s2}\prod_{u\neq r,s}\delta_{l_u0}+\mathcal{O}(\lambda^2),\end{aligned} \tag{4.10}$$

from which we get

$$\left[T^{l_0 l_1 \ldots l_N}_{10\ldots 0}(\lambda)\right]^2 = \sum_{r=0}^{N} \left(t_0^r\right)^2 \delta_{l_r 1} \prod_{s\neq r} \delta_{l_s 0} + \mathcal{O}(\lambda^2). \tag{4.11}$$

Replacing equation (4.11) in equation (4.5) we obtain for $\mathcal{A}^{10\ldots 0}_{10\ldots 0}(t)$, which we denote as $f_{00}(t;\lambda)$,

$$f_{00}(t;\lambda) = \exp\left(-\frac{\mathrm{i}t}{2}\sum_{r=0}^{N}\Omega_r\right)\sum_{r=0}^{N}\left(t_0^r\right)^2 \exp\left\{-\mathrm{i}t\left[E_{1_r}(\lambda) - \frac{\Omega_r}{2}\right]\right\} + \mathcal{O}(\lambda^2), \tag{4.12}$$

where we have from the appendix $E_{1_r}(\lambda) = \frac{3}{2}\Omega_r + \frac{15}{4}\lambda\Omega_r + \mathcal{O}(\lambda^2)$ and substituting in equation (4.12) we get

$$f_{00}(t;\lambda) = \sum_{r=0}^{N}\left(t_0^r\right)^2\left(1 - \frac{15}{4}\mathrm{i}\lambda t\Omega_r\right)\mathrm{e}^{-\mathrm{i}t\Omega_r} + \mathcal{O}(\lambda^2), \tag{4.13}$$

where we have discarded a phase factor that does not contribute to the probability. From the above equation, we get the probability for the particle oscillator to remain in the first excited level,

$$|f_{00}(t;\lambda)|^2 = |f_{00}(t)|^2 + \frac{15\lambda t}{4}\frac{\partial}{\partial t}|f_{00}(t)|^2 + \mathcal{O}(\lambda^2), \tag{4.14}$$

where

$$f_{00}(t) = \sum_{r=0}^{N}\left(t_0^r\right)^2 \mathrm{e}^{-\mathrm{i}\Omega_r t}. \tag{4.15}$$

Equation (4.14) is the same as the one obtained in [8]. In this reference, it has been supposed that the dressed coordinates are not affected by the nonlinear interactions and then the only correction for the probability amplitude comes from the corrections to the first excited eigenenergies, that is, we assumed equation (4.12) as our starting point. We obtained the same result because at order λ the square of $T^{l_0 l_1 \ldots l_N}_{10\ldots 0}$ is given only by the square of the first term in equation (4.10), that does not depend on λ.

To see the effect of the last term in equation (4.14) consider the situation in which the frequencies ω_k are continuously distributed. In this case, as shown in [1], the first term in equation (4.14), the probability for the particle oscillator to remain in the first excited level in the absence of nonlinearities, $|f_{00}(t)|^2$, is a decreasing, almost exponentially, function of time. Then, the last term is a negative quantity, from which we can conclude that the effect of the nonlinearity, at first order in the coupling constant, is the enhancement of the decay of the first excited level of the particle oscillator.

5. Conclusions

In this paper, after clarifying what we understand by dressed coordinates and dressed states, we have developed a formal method to construct perturbatively dressed coordinates in nonlinear systems. Although we restricted our calculations to a very special quartic nonlinear term, we have pointed out the necessity of factoring a term in order to avoid artificial singularities which otherwise would appear if we do not make such factorization. That this factorization is the correct one has been checked by using an exactly solvable sextic nonlinear model. Then, in more general nonlinear systems, one can follow the same procedure to construct dressed coordinates.

At the end of section 2, we remarked that for nonlinear systems, in the number representation, the dressed ground state is not equivalent to the ground state of the total system, see equation (2.18). This fact must not be seen as in contradiction with our definition of dressed coordinates, since we have defined them by requiring the equivalence of the dressed ground state in dressed coordinates representation and the ground state of the system in normal coordinates representation. We can understand the mentioned non-equivalence by noting that although the dressed ground state is an eigenstate of the dressed number operators (associated with the dressed coordinates) the ground state of the system, in general, is not an eigenstate of the collective number operators. For example, in the quartic nonlinear case, one can easily verify that the ground state of the system is not an eigenstate of collective number operators, but a linear superposition of eigenstates of these operators (see the appendix, equation (A.5)).

Finally, we considered the computation of the probability of the particle oscillator to remain excited in the first excited level, and showed that our result coincides with that obtained in [8] at first order in the coupling constant. However, at higher orders in the coupling constant the treatment given in [8] will give incorrect results, since equation (4.12) is only valid at first order in the coupling constant.

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Appendix A. The perturbed eigenfunctions and eigenvalues

It is easy to see that the eigenfunctions of the quartic anharmonic oscillator can be written formally as those given in equation (3.4). We have to note only that any wavefunction can be expanded in the basis $\phi_n(Q)$ (we omit here the index r), given by the eigenvalues of the linear part of the Hamiltonian. And since $\phi_n(Q)$ are given by $\exp(-\Omega Q^2/2)$ times a Hermite polynomial of degree n, we see that an expression of the type given in equation (3.4) follows. In what follows, we compute $G_n^{(1)}(\sqrt{\Omega}Q)$ and $G_n^{(2)}(\sqrt{\Omega}Q)$ by using standard perturbation theory.

At second order in standard perturbation theory, the eigenfunctions and eigenvalues of a Hamiltonian $\hat{H} = \hat{H}_0 + \lambda\hat{V}$ are given, respectively, by

$$|n,\lambda\rangle = |n\rangle + \lambda\sum_{k\neq n}\frac{V_{kn}|k\rangle}{E_n - E_k} + \lambda^2\left(\sum_{k,l\neq n}\frac{V_{kl}V_{ln}|k\rangle}{(E_n-E_k)(E_n-E_l)} - V_{nn}\sum_{k\neq n}\frac{V_{kn}|k\rangle}{(E_n-E_k)^2}\right) + \mathcal{O}(\lambda^3) \tag{A.1}$$

and

$$E_n(\lambda) = E_n + \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n - E_k} + \mathcal{O}(\lambda^3), \tag{A.2}$$

where

$$V_{kn} = \langle k|\hat{V}|n\rangle \tag{A.3}$$

and $|n\rangle$ and E_n are, respectively, eigenfunctions and eigenvalues of the unperturbed Hamiltonian $\hat{H}_0$.

For the anharmonic oscillator with $\hat{V} = \Omega^3 \hat{Q}^4$, we easily obtain

$$V_{kn} = \frac{\Omega}{4}[\sqrt{k_4}\delta_{k,n-4} + 2(2n-1)\sqrt{k_2}\delta_{k,n-2} + 3(2n^2+2n+1)\delta_{k,n} + 2(2n+3)\sqrt{n_2}\delta_{k,n+2} + \sqrt{n_4}\delta_{k,n+4}], \tag{A.4}$$

where $k_n = (k+1)(k+2)\cdots(k+n)$. Replacing equation (A.4) in equations (A.1) and (A.2) we obtain, respectively,

$$|n,\lambda\rangle = |n\rangle + \lambda(a_n'|n-4\rangle + b_n'|n-2\rangle + c_n'|n+2\rangle + d_n'|n+4\rangle) + \lambda^2(e_n'|n-8\rangle + f_n'|n-6\rangle + g_n'|n-4\rangle + h_n'|n-2\rangle + t_n'|n+2\rangle + u_n'|n+4\rangle + v_n'|n+6\rangle + w_n'|n+8\rangle) + \mathcal{O}(\lambda^3) \tag{A.5}$$

and

$$E_n(\lambda) = \left(n+\frac{1}{2}\right)\Omega + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \mathcal{O}(\lambda^3), \tag{A.6}$$

where

$$a_n' = \frac{1}{16}\sqrt{(n-4)_4}, \qquad b_n' = \frac{(2n-1)}{4}\sqrt{(n-2)_2}, \qquad c_n' = -\frac{(2n+3)}{4}\sqrt{n_2}, \qquad d_n' = -\frac{1}{16}\sqrt{n_4}; \tag{A.7}$$

$$e_n' = \frac{1}{512}\sqrt{(n-8)_8}, \qquad f_n' = \frac{(6n-11)}{192}\sqrt{(n-6)_6},$$
$$g_n' = \frac{1}{16}(2n^2-9n+7)\sqrt{(n-4)_4}, \qquad h_n' = -\frac{1}{64}(2n^3+93n^2-107n+66)\sqrt{(n-2)_2},$$
$$t_n' = -\frac{1}{64}(2n^3-123n^2-359n-300)\sqrt{n_2}, \qquad u_n' = \frac{1}{16}(2n^2+13n+18)\sqrt{n_4},$$
$$v_n' = \frac{(6n+17)}{192}\sqrt{n_6}, \qquad w_n' = \frac{1}{512}\sqrt{n_8}; \tag{A.8}$$

$$E_n^{(1)} = \frac{3}{4}(2n^2+2n+1)\Omega \tag{A.9}$$

and

$$E_n^{(2)} = -\tfrac{1}{8}(34n^3+51n^2+59n+21)\Omega. \tag{A.10}$$

Writing equation (A.5), in coordinate representation, in the form given in equation (3.4) we get for $G_n^{(1)}(\sqrt{\Omega}Q)$ and $G_n^{(2)}(\sqrt{\Omega}Q)$, respectively,

$$G_n^{(1)}(\xi) = a_n H_{n-4}(\xi) + b_n H_{n-2}(\xi) + c_n H_{n+2}(\xi) + d_n H_{n+4}(\xi) \tag{A.11}$$

and

$$G_n^{(2)}(\xi) = e_n H_{n-8}(\xi) + f_n H_{n-6}(\xi) + g_n H_{n-4}(\xi) + h_n H_{n-2}(\xi) + t_n H_{n+2}(\xi) + u_n H_{n+4}(\xi) + v_n H_{n+6}(\xi) + w_n H_{n+8}(\xi), \tag{A.12}$$

where $\xi = \sqrt{\Omega}Q$,

$$a_n = \frac{a_n'}{\sqrt{2^{n-4}(n-4)!}}, \qquad b_n = \frac{b_n'}{\sqrt{2^{n-2}(n-2)!}}, \qquad c_n = \frac{c_n'}{\sqrt{2^{n+2}(n+2)!}}, \qquad d_n = \frac{d_n'}{\sqrt{2^{n+4}(n+4)!}} \tag{A.13}$$

and

$$
\begin{aligned}
e_n &= \frac{e'_n}{\sqrt{2^{n-8}(n-8)!}}, & f_n &= \frac{f'_n}{\sqrt{2^{n-6}(n-6)!}}, \\
g_n &= \frac{g'_n}{\sqrt{2^{n-4}(n-4)!}}, & h_n &= \frac{h'_n}{\sqrt{2^{n-2}(n-2)!}}, \\
t_n &= \frac{t'_n}{\sqrt{2^{n+2}(n+2)!}}, & u_n &= \frac{u'_n}{\sqrt{2^{n+4}(n+4)!}}, \\
v_n &= \frac{v'_n}{\sqrt{2^{n+6}(n+6)!}}, & w_n &= \frac{w'_n}{\sqrt{2^{n+8}(n+8)!}}.
\end{aligned}
\tag{A.14}
$$

References

[1] Andion N P, Malbouisson A P C and Mattos Neto A 2001 *J. Phys. A: Math. Gen.* **34** 3735
[2] Flores-Hidalgo G, Malbouisson A P C and Milla Y W 2002 *Phys. Rev.* A **65** 063414 (*Preprint* physics/0111042)
[3] Flores-Hidalgo G and Malbouisson A P C 2002 *Phys. Rev.* A **66** 042118 (*Preprint* quant-ph/0205042)
[4] Casana R, Flores-Hidalgo G and Pimentel B M 2005 *Phys. Lett.* A **337** 1 (*Preprint* physics/0410063)
[5] Flores-Hidalgo G and Malbouisson A P C 2005 *Phys. Lett.* A **337** 37 (*Preprint* physics/0312003)
[6] Hulet R G, Hilfer E S and Kleppner D 1985 *Phys. Rev. Lett.* **55** 2137
[7] Jhe W, Anderson A, Hinds E A, Meschede D, Moi L and Haroche S 1987 *Phys. Rev. Lett.* **58** 666
[8] Flores-Hidalgo G and Malbouisson A P C 2003 *Phys. Lett.* A **311** 82 (*Preprint* physics/0211123)
[9] Thirring W and Schwabl F 1964 *Ergeb. Exakt. Naturw.* **36** 219
[10] Weiss U 1993 *Quantum Dissipative Systems* (Singapore: World Scientific)
[11] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[12] Walls D F and Milburn G J 1994 *Quantum Optics* (Berlin: Springer)
[13] Ford G W, Lewis J T and O'Connell R F 1988 *J. Stat. Phys.* **53** 39
[14] Flores-Hidalgo G and Ramos R O 2003 *Physica* A **326** 159 (*Preprint* hep-th/0206022)
[15] Ordonez G and Kim S 2004 *Phys. Rev.* A **70** 032702
[16] Polonsky N 1964 *Doctoral Thesis* Ecole Normale Supérieure, Paris
[17] Haroche S 1964 *Doctoral Thesis* Ecole Normale Supérieure, Paris
[18] Cohen-Tannoudji C, Dupont-Roc J and Grynberg G 1992 *Atom–Photon Interactions* (New York: Wiley)
[19] Cohen-Tannoudji C 1994 *Atoms in Electromagnetic Fields* (Singapore: World Scientific)
[20] Skála L, Cízek J, Dvorák J and Špirko V 1996 *Phys. Rev.* A **53** 2009